

A LORENTZ COVARIANT NONCOMMUTATIVE GEOMETRY

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ABSTRACT. A noncommutative geometry that preserves lorentz covariance was introduced by Hartland Snyder in 1947. We show that this geometry has unusual properties under momentum translation, and derive for it a form of star product. This work was presented at the 7th International Wigner Symposium, August 2001, College Park, MD, USA.

1. INTRODUCTION

The possibility that space time may be noncommutative in the sense that:

$$(1) \quad [x^\mu, x^\nu] = i\theta^{\mu\nu}$$

where $\theta^{\mu\nu}$ is a constant, has appeared recently in string theory, [1] and also in measurement theory. [2] The $\theta^{\mu\nu}$ may be thought of as a kind of background field, and its presence implies some small violation of Lorentz invariance. The possibility of detecting this Lorentz violation has been discussed in Ref. [3].

It follows from Eq.(1) that:

$$(2) \quad \Delta x^\mu \Delta x^\nu \geq |\theta^{\mu\nu}|$$

and $\theta^{\mu\nu}$ provides some sort of minimal length. There are measurement theory arguments that suggest that $\theta^{\mu\nu}$ should be of the order of the square of the Planck length ℓ_P^2 . [2]

In 1947, Hartland S. Snyder [4] suggested that a minimal length might be a natural way of introducing a high momentum cut-off into field theory calculations. To this end, he introduced a Lorentz covariant noncommutativity, which, in modern notation, takes the form:

$$(3) \quad [x^\mu, x^\nu] = i\theta M^{\mu\nu}$$

where here θ is a constant, and the $M^{\mu\nu}$ are the generators of the Lorentz transformations, satisfying:

$$(4) \quad [M^{\mu\nu}, M^{\rho\sigma}] = i[\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\nu\sigma} M^{\mu\rho}]$$

$$[M^{\mu\nu}, x^\rho] = i[\eta^{\mu\rho} x^\nu - \eta^{\nu\rho} x^\mu]$$

$$[M^{\mu\nu}, P^\rho] = i[\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu]$$

where $\eta^{\mu\nu}$ is the metric tensor, = (- , + , + , +) , and P^ρ is the momentum 4-vector, assumed to satisfy:

$$(5) \quad [P^\mu, P^\nu] = 0$$

and, to be consistent with the Jacobi identity, the canonical x - P commutation relation must be replaced by:

$$(6) \quad [x^\mu, P^\nu] = i[\eta^{\mu\nu} + \theta P^\mu P^\nu]$$

2. SNYDERS REPRESENTATIONS

Snyder found two representations for this space. The first representation works for positive θ . It was based on a 5 dimensional Minkowski manifold, $\{\eta^m, m = 0,..4\}$, with metric $g_{\mu\nu} = \eta_{\mu\nu}$, $g_{\mu 4} = 0$, $\mu, \nu = 0,..3$, $g_{44} = +1$. The coordinate operators x^μ he identified with certain of the generators M^{mn} , $m,n = 0,..4$, of $\text{SO}(1,5)$, the Lorentz group acting on this manifold:

$$(7) \quad x^\mu = aM^{\mu 4}$$

where $a^2 = \theta$.

For the momentum operators he found:

$$(8) \quad P^\mu = \frac{\eta^\mu}{a\eta^4}$$

In this representation, it can be seen that the x^i and the M^{ij} , with $i, j = 1,2,3$, are rotation operators for the four space-like coordinates, thus they generate $\text{SO}(4)$, and it follows that the x^i have discrete eigenvalues that are multiples of a . The time coordinate, x^0 , is a boost operator in the η^4 direction, and therefore has continuous eigenvalues.

Snyder's second representation works for both positive and negative θ . In modern notation, it is based on ordinary canonical 4-vector position operators ξ^μ , obeying the commutation relations:

$$(9) \quad [\xi^\mu, \xi^\nu] = 0$$

$$[\xi^\mu, P^\nu] = i\eta^{\mu\nu}$$

and, defining a dilation operator as:

$$(10) \quad D = \frac{1}{2}\{\xi^\alpha, P_\alpha\}$$

then Snyder's coordinate operators can be written as:

$$(11) \quad x^\alpha = \xi^\alpha + \frac{\theta}{2}\{P^\alpha, D\}$$

3. TRANSLATIONS IN MOMENTUM SPACE

In the following, we will use a dot to indicate the 4-vector inner product, and hatted symbols to denote operators. The operators

$$(12) \quad U_0(k) = e^{ik \cdot \hat{\xi}}$$

perform translations in momentum space,

$$(13) \quad U_0(k)|q^\alpha\rangle = |q^\alpha + k^\alpha\rangle$$

where $|q\rangle$ is an eigenket of the 4-momentum operator. The corresponding operators in the Snyder geometry:

$$(14) \quad U(k) = e^{ik \cdot \hat{x}}$$

do not act as simply. On a momentum eigenket they give:

$$(15) \quad U(k)|q^\alpha\rangle = A|q^\alpha g + k^\alpha f\rangle$$

where A, g and f are functions of k^2 and $k \cdot q$:

$$(16) \quad g = \left[\cos(\phi) - k \cdot q \sqrt{\frac{\theta}{k^2}} \sin(\phi) \right]^{-1}$$

$$(17) \quad f = \left[k \cdot q (\cos(\phi) - 1) + \sqrt{\frac{k^2}{\theta}} \sin(\phi) \right] g/k^2$$

$$(18) \quad A = \left| \det \frac{\partial(q^\alpha g + k^\alpha f)}{\partial q^\beta} \right|^{1/2}$$

here

$$(19) \quad \phi = \sqrt{\theta k^2}$$

To first order in θ , these become:

$$(20) \quad g = 1 + \theta \left(\frac{k^2}{2} + k \cdot q \right)$$

$$(21) \quad f = 1 + \theta \left(\frac{k^2}{3} + \frac{k \cdot q}{2} \right)$$

$$(22) \quad A = 1 + \frac{5}{4}\theta (k^2 + 2k \cdot q)$$

Let us denote the transformed momentum, $q^\alpha g + k^\alpha f$, by \dot{q}^α . The transformation has the strange property that \dot{q}^α becomes infinite for those momenta k^α that satisfy:

$$(23) \quad \cot\left(\sqrt{\theta k^2}\right) = k \cdot q \sqrt{\frac{\theta}{k^2}}$$

This is simplest when the initial momentum q^α is zero. Then \dot{q}^α becomes infinite when

$$(24) \quad k^2 = \frac{(n\pi)^2}{\theta}$$

for any integer n. If we take θ as positive, these are spacelike momenta, and a negative θ would give us infinities at timelike k. The measurement theory arguments, with θ proportional to ℓ_P^2 , imply that these are multiples of the Planck mass.

4. THE STAR PRODUCT

For operators that satisfy Eq. (1) with a constant matrix, there is a one to one correspondence between functions f on Minkowski space and operators F on Hilbert space, $F \iff f$, given by:

$$(25) \quad F(\hat{x}) = \int \frac{d^4 k}{(2\pi)^2} e^{ik \cdot \hat{x}} \tilde{f}(k)$$

where \tilde{f} is related to a function f in position space:

$$(26) \quad \tilde{f}(k) = \int \frac{d^4 x}{(2\pi)^2} e^{-ik \cdot x} f(x)$$

and also can be related to the operator F:

$$(27) \quad \tilde{f}(k) = \text{Tr} (e^{-ik \cdot \hat{x}} F(\hat{x}))$$

where the Tr is defined as:

$$(28) \quad \text{Tr} (A) = \text{Lim}_{\Lambda \rightarrow \infty} \frac{(2\pi)^2}{\Lambda^4} \int^{\Lambda} d^4 q \langle q | A | q \rangle$$

where the $|q\rangle$ are momentum eigenkets. This works because

$$(29) \quad \langle h | e^{-iq \cdot \hat{x}} e^{-ik \cdot \hat{x}} | h \rangle = \delta(q - k)$$

Given a second operator $G \iff g$, we have for the product:

$$(30) \quad FG \iff f \star g$$

where the star product is defined as:

$$(31) \quad f \star g = \text{Lim}_{x' \rightarrow x} e^{\frac{i}{2}\theta^{\alpha\beta}\partial_{\alpha}\partial'_{\beta}} f(x)g(x')$$

$$(32) \quad = f(x)g(x) + \frac{i}{2}\theta^{\alpha\beta}\partial_{\alpha}f(x)\partial_{\beta}g(x)$$

to first order in θ .

5. THE SNYDER STAR PRODUCT

For a function f , using the Snyder operators \hat{x} it is certainly possible to define a Hilbert space operator F as is done in Eqs.(25) and (26). That is, $f \implies F$. However Eqs.(27) and (28) cannot be used to find f given F . This is because the non-linearity of the momentum transformation shown in Eqs.(15) to (19) gives, instead of Eq.(29),

$$(33) \quad \langle h | e^{-iq \cdot \hat{x}} e^{-ik \cdot \hat{x}} | h \rangle = \delta [hg(q, h) + qf(q, h) - hg(k, h) - kf(k, h)]$$

and integrating this over all h yields an ordinary function of q and k , instead of the required dirac delta function.

Nevertheless, it is possible to go from F to f . Let $|0\rangle$ denote the zero eigenket of the momentum operator. It can be shown that, to first order in θ ,

$$(34) \quad \langle 0 | e^{-iq \cdot \hat{x}} e^{-ik \cdot \hat{x}} | 0 \rangle = \left(1 + \frac{\theta}{2}q^2\right) \delta(q - k)$$

from which it follows that

$$(35) \quad \tilde{f}(k) = (2\pi)^2 \left(1 - \frac{\theta}{2}k^2\right) \langle 0 | e^{-ik \cdot \hat{x}} F(\hat{x}) | 0 \rangle$$

and we also have $F \implies f$, for such functions F of \hat{x} alone.

More generally, for any operator A , we could consider the function $f(x)$ given by

$$(36) \quad f(x) = \int d^4 k e^{ik \cdot x} \left(1 - \frac{\theta}{2}k^2\right) \langle 0 | e^{-ik \cdot \hat{x}} A | 0 \rangle$$

A problem arises when we consider products of functions of the \hat{x}^{α} , because the commutation rules (3) do not close on the \hat{x}^{α} , but involve the $M^{\alpha\beta}$. A product of functions F of \hat{x}^{α} becomes a function of both the \hat{x}^{α} and the $M^{\alpha\beta}$. Nevertheless, we

can still use Eq.(36) to find a function of x^α associated with the operator product. Let $f \Rightarrow F$ and $g \Rightarrow G$. Defining

$$(37) \quad \tilde{f}g(k) = (2\pi)^2 \left(1 - \frac{\theta}{2}k^2\right) \langle 0 | e^{-ik \cdot \hat{x}} F(\hat{x}) G(\hat{x}) | 0 \rangle$$

and $f \star g(x)$ as the Fourier transform of $\tilde{f}g(k)$, we get, to first order in θ , the Snyder star product:

$$(38) \quad f \star g(x) = f(x)g(x) - \text{Lim}_{u,v \rightarrow x} \frac{\theta}{2} x \cdot \left[\nabla_v \left(\frac{\nabla_u^2}{2} + \nabla_u \cdot \nabla_v \right) - \nabla_u \left(\nabla_v^2 + \frac{\nabla_u \cdot \nabla_v}{2} \right) \right] f(u)g(v)$$

The operators \hat{x}^α and $M^{\alpha\beta}$ generate an algebra acting on a Hilbert space H . Eq.(36) is a map from operators on H to a Snyder \star algebra of functions on Minkowski space. It takes multiples of the operators $e^{\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}}$ into the identity since, to first order in θ

$$(39) \quad e^{ih \cdot \hat{x}} e^{ik \cdot \hat{x}} = e^{ip \cdot \hat{x}} e^{\frac{-i}{2}h \cdot M \cdot k}$$

where

$$(40) \quad p^\alpha = h^\alpha + k^\alpha + \frac{\theta}{3} \left[k^\alpha \left(\frac{h^2}{2} + h \cdot k \right) - h^\alpha \left(k^2 + \frac{h \cdot k}{2} \right) \right]$$

and the state $|0\rangle$, being invariant under Lorentz transformations, takes the operators $M^{\alpha\beta}$ into zero.

6. DISCUSSION

The presence of a constant matrix $\theta^{\mu\nu}$ in the commutation relation Eq.(1) spoils not only Lorentz invariance, but also the isotropy of empty space-time. The Snyder model, although it preserves Lorentz invariance, still violates spatial isotropy. One sign of this is the different spectra for space and time coordinates. Another is the presence of the coordinate x in Eq.(38), the Snyder star product. This means the point $x = 0$ is a preferred point, which is not surprising, as rotations and boosts leave this point invariant.

Eq.(36) gives a map from an algebra of operators on Hilbert space to the algebra of functions on coordinate space furnished with the Snyder \star product. The map cannot be an isomorphism, since some operators go into the identity. It is not known if the map is some kind of homomorphism, in particular, it is not known if the Snyder \star product is associative.

Is the Snyder algebra relevant? This is a noncommutative geometry based on $\text{SO}(4,1)$. A similar geometry based on $\text{SO}(5)$ has recently been investigated in Ref. [5].

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